Resource Based Cooperative Games: Optimization, Fairness and Stability

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Abstract

We introduce the class of *resource-based coalitional games*, a novel and complete representation of cooperative games extending *threshold task games* introduced by Chalkiadakis *et al.*. Starting from the class of weighted voting games (the simplest example of resource-based coalitional games), we provide efficient algorithms which compute solution concepts for resource-based coalitional games; these include approximately optimal coalition structures and the Shapley value. We also present non-trivial bounds on the cost of stability for this class; in particular, we improve upon the bounds given in Bachrach *et al.* for weighted voting games.

1 Introduction

Cooperative game theory studies situations in which agents or players have the possibility of forming coalitions and sharing revenue. Efficiently computing solution concepts for cooperative games is the focus of several research papers; such solution concepts include finding an optimal coalition structure, computing a stable (core) payoff division, and computing the Shapley value. All three problems are notoriously difficult to compute for general cooperative games; this is often due to the fact that general cooperative games require an exponential number of bits to represent: one must encode the value of every subset of players. Thus, several works focus on finding *succinct, representations of cooperative games*: these are classes of cooperative games which can be represented using a polynomial number of bits; however, some of these classes can represent general cooperative games (using an exponential number of bits) [14, 19, 28, 3].

We propose a novel, yet intuitive, complete representation of coalitional games based on the notion of *tasks* and *resources*, which we term r *Threshold Task Games* (r-TTGs); here, r is the number of resource types available. Briefly, we are given a list of tasks that can be completed by players, each requiring different quantities of resources to complete. The value of a coalition is the value of the best task it can complete given its available resources. r-TTGs include the canonical class of *weighted voting games* (WVGs), as well as the more recently introduced class of *threshold task games* (TTGs) [10]: these are games where all tasks require a single resource. Another example of resource-based games are vector weighted voting games, introduced by Elkind *et al.*. However, other natural scenarios can be captured by resource-based games: consider, for example, a set of computational tasks, each requiring a different amount of computational resources (e.g. RAM, GPU cycles, hard-drive space etc.); our players are computational agents who try to complete these tasks using their available resources.

1.1 Our Contribution

We introduce a resource-based class of coalitional games which we call r *Threshold Task Games* (r-TTGs), and provide several efficient algorithms computing solution concepts for this class. We begin by providing efficient approximation algorithms for the optimal coalition structure problem for r-TTGs. In more detail, we provide a $\frac{1}{2}$-approximation algorithm for the single resource case (i.e. TTGs), as well as a bound for the general case, parameterized by the number of resource types r. Next, we bound the cost of stability for r-TTGs: this is the minimal payoff ensuring that no coalition can deviate. In particular, we improve upon the bounds provided by Bachrach *et al.* for weighted voting games. Finally, we present a dynamic programming based algorithm computing the Shapley value for r resource-based coalitional games, generalizing the algorithm for WVGs in [23] (see [11, Chapter 4]).

1.2 Related Work

Weighted voting games are an extremely well-studied class of games: on the one hand, they are computationally succinct (requiring only n weights and a threshold to describe); on the other hand, computing solution concepts for weighted voting games is well-known to be computationally intractable. One of the best studied solution concepts for WVGs are power indices, such as the Shapley value [27] and the Banzhaf index [6]; the complexity of computing the Shapley value in WVGs is analyzed in [23]. Computational aspects of weighted voting games were explored by Elkind *et al.*, who establishes the computational intractability of computing outcomes in the least core and the nucleolus of a WVG. Elkind *et al.* establish the intractability of computing coalition structures for WVGs. The complexity of solution concepts for general cooperative games has been well studied in the literature, dating back to Deng and Papadimitriou; more recent works include [7, 12, 20, 21] (see Chalkiadakis *et al.* and Chalkiadakis and Wooldridge for an overview).

Threshold task games were introduced by Chalkiadakis *et al.*; their work departs from the classic cooperative game model, allowing agents to split resources amongst several tasks. The only work we are aware of that studies a TTG model in the classic cooperative game setting is by Balcan *et al.*, who establish the PAC learnability of TTGs. The optimal coalition structure generation problem is also well-studied (we refer our reader to [25] for a recent overview); other related works include [3, 4]. A model similar to the TTG setting where tasks are limited in supply is studied by Anshelevich and Sekar.

Bachrach *et al.* introduce and study the *cost of stability* (CoS). Bachrach *et al.* bound the CoS for WVGs; however, they assume that coalition structures do not form, in which case the cost of stability is rather high. The cost of stability has been studied in several other works, making either assumptions on the class of cooperative games [26], or on the underlying agent interaction structure [8, 24]

Several works study efficient representations of coalitional games; these include the seminal work by Ieong and Shoham on MC-nets, as well as skill-based [3], type-based [28], and synergy-based [14] representations (see [11, Chapter 3] for an overview).

2 Preliminaries

A *cooperative game* $G = \langle N, v\rangle$ consists of a set of agents $N = \{1, \dots, n\}$ and a function $v : 2^N \to \mathbb{R}$; v is called the *characteristic function*. Given a set of players S (also known as a *coalition*), $v(S)$ is the value of S; we assume that $v(\emptyset) = 0$, and that v is monotone: for every two coalitions $S \subseteq T \subseteq N$, we have $v(S) \le v(T)$.

Given a cooperative game $G = \langle N, v\rangle$, a partition of players into coalitions is called a *coalition structure*. We say that a coalition structure CS^* is *optimal* if it maximizes social welfare; that is,

$$CS^* \in argmax \left\{ \sum_{C \in CS} v(C) : CS \text{ is a partition of } N \right\}.$$

We let $OPT(G)$ be the value of an optimal coalition structure over G. We refer to the problem of finding an optimal coalition structure (also known as the *coalition structure generation problem*) as OPTCS.

Once players have formed a coalition structure, they need to divide the revenue in some reasonable manner. Given a coalition structure CS, an *imputation* for CS is a vector $\vec{x} \in \mathbb{R}^n_+$ satisfying $\sum_{i \in C} x_i = v(C)$ for all $C \in CS$; that is, the total amount paid out to a coalition C (also written as

$x(C)$) must equal its value. The tuple $\langle CS, \vec{x} \rangle$ is called an *outcome* of G. Let us denote by $\mathcal{I}(G)$ the set of all outcomes for G.

The core is a subset of outcomes in $\mathcal{I}(G)$ from which no coalition can deviate; that is,

$$Core(G) = \{\langle CS, \vec{x} \rangle \in \mathcal{I}(G) : x(C) \geq v(C), \forall C \subseteq N\}.$$

We observe that if $\langle CS, \vec{x} \rangle \in Core(G)$ then CS must be optimal (else at least one coalition C belonging to an optimal coalition structure CS^* can deviate). Unfortunately, the core of a game can be empty.

Example 2.1. Consider a 3 player game where

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

The optimal coalition structure has a value of 1. However, it is easy to check that if every coalition of size 2 receives a payoff of at least 1, the total payoff to all players must be at least $\frac{3}{2}$.

As Example 2.1 implies, if one wishes to stabilize a game, it may require an external source of revenue that pays players an additional sum. The minimal amount needed can be thought of as a solution to the following linear program

$$\begin{aligned} &\min \sum_{i \in N} x_i && (1) \\ &\text{s.t. } x(C) \geq v(C) && \forall C \subseteq N. \end{aligned}$$

Let V^* be the value of the optimal solution to (1); the *cost of stability* of a game G is then defined by the ratio between V^* and $OPT(G)$.

$$CoS(G) = \frac{V^*}{OPT(G)}.$$

Another solution concept that captures the notion of fairness is the Shapley-Shubik power index [27]. Given a permutation $\sigma : N \to N$, let $P_i(\sigma) = \{j \in N : \sigma(j) < \sigma(i)\}$ and $m_i(\sigma) = v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma))$. The Shapley value of agent i is the expected marginal contribution of i to a permutation chosen uniformly at random. Formally,

$$\phi_i(G) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma),$$

where $\Pi(N)$ denotes the set of all permutations of N.

3 Resource-Based Cooperative Games

Weighted voting games (WVG) are the simplest class of resource-based cooperative games: we are given a set of players N; each player i has a weight $w_i \in \mathbb{Z}_+$, and a threshold $q \in \mathbb{Z}_+$ called quota, then:

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geq q \\ 0 & \text{otherwise,} \end{cases}$$

where we denote $w(S) = \sum_{i \in S} w_i$. Informally, a coalition wins if its total weight is greater than q, and loses otherwise. It is sometimes useful to assume that the value of winning coalitions is not 1, but rather some arbitrary value $\alpha > 0$: $v(S) = \alpha$ if $w(S) \geq q$ and is 0 otherwise. *Threshold task* *games* (TTG) generalize WVGs in the sense that there are multiple thresholds and each coalition gains the best value possible. In other words, the value of a coalition is the value of the best task that it can complete with its resources. Formally, a TTG is given by a set of players N where each player has a weight $w_i \in \mathbb{Z}_+$; in addition, we have a set of tasks $\mathcal{T} = \{t_1, \dots, t_m\}$, where each task $t_j \in \mathcal{T}$ has a threshold $q_j \in \mathbb{Z}_+$ and a value $v_j \in \mathbb{Z}_+$. The characteristic function v is defined as follows:

$$v(S) = \max_{j \in [m]} \{v_j : w(S) \geq q_j\}.$$

By this definition, we assume that there is potentially an infinite supply of each task (thus, if a coalition S completes the task t_j, other coalitions are free to complete it as well); however, an infinite supply of tasks is not strictly necessary: when forming a coalition structure, at most n coalitions can execute any given task $t_j \in \mathcal{T}$. Therefore, given a set of tasks $\mathcal{T}$, we expand $\mathcal{T}$ so that every task $t_j \in \mathcal{T}$ has at least n copies, and that in any coalition structure, every task $t_j \in \mathcal{T}$ is completed by at most one coalition. Under this extension, we say that $t_j, t_k \in \mathcal{T}$ are of the same type if they have the same threshold and value. We can also safely assume that if two tasks t_j and t_k have $q_j \leq q_k$ then $v_j \leq v_k$; otherwise no coalition will ever complete t_k. Finally, we assume that all tasks in $\mathcal{T}$ can be completed; that is, for every $t_j \in \mathcal{T}$, $q_j \leq w(N)$.

We further generalize this definition as follows: suppose that instead of having a single resource at their disposal, players have r different resources; each task requires a certain amount of each resource to be completed, and the value of a coalition S is the value of the best task it can complete. More formally, v is an r-TTG if each player owns a resource vector $\vec{w}_i \in \mathbb{R}^r_+$; there is a set of tasks $\mathcal{T}$, where every task $t_j \in \mathcal{T}$ has a value v_j and a threshold vector $\vec{q}_j \in \mathbb{R}^r_+$. The value of a coalition $S \subseteq N$ is given by

$$v(S) = \max_{j \in [m]} \left\{ v_j : \sum_{i \in S} \vec{w}_i \geq \vec{q}_j \right\}.$$

An r-WVG [17] is a subclass of r-TTGs with a single task (whose value is 1).

Generally, we say that a game is r *resource-based* if there exists some list of vectors $\vec{w}_1, \dots, \vec{w}_n \in \mathbb{R}^r_+$ such that $v(S)$ is only a function of $\sum_{i \in S} \vec{w}_i$. More formally: there exists some function $f : \mathbb{R}^r_+ \to \mathbb{R}_+$ such that $v(S) = f(\sum_{i \in S} \vec{w}_i)$ for all $S \subseteq N$. TTGs are 1 resource based, but other games fit this definition as well: the game $v(S) = \log(|S|)$ is 1 resource based with $w_i = 1$ for all $i \in N$. We say that v is a monotone r resource-based game if f is an increasing function with respect to each of its dimensions. The following proposition states that all r resource-based games can be modeled as r-TTGs, with a potentially exponential number of task types.

Proposition 3.1. *Given a monotone r resource-based game v, there exists an r-TTG v' such that $v(S) = v'(S)$ for all $S \subseteq N$; v' may have exponentially many task types.*

Proof. Since v is a monotone r resource-based game, there exists some a set of n resource vectors $\vec{w}_i \in \mathbb{R}^r_+$ and some increasing function $f : \mathbb{R}^r \to \mathbb{R}$ such that $v(S) = f(\vec{w}(S))$ for all $S \subseteq N$. Let $V = \{f(\vec{w}(S)) \mid S \subseteq N\}$ be the set of values that f takes over subsets of N, and assume that $V = \{v_1, \dots, v_k\}$. Let $Q = \{\vec{q} : \exists S \subseteq N, \vec{w}(S) = \vec{q}\}$. For every value $v_j \in V$, let $Q_j \subset Q$ be the set of vectors $\vec{q} = (q^1, \dots, q^r)$ satisfying that $f(\vec{q}) = v_j$ and that for every $\vec{q'} \in Q$ with $f(\vec{q'}) = v_j$ there exists some $e \in [r]$ such that $q'^e > q^e$. The r-TTG defined with tasks $t_j = \langle \vec{q}, v_j \rangle$ for all $\vec{q} \in Q_j$ and for all $j \in [k]$ and player weights as per $\vec{w}_i$ for $i \in [n]$ (denoted v') has the same values as v on all coalitions. Indeed, for every set $S \subseteq N$, there exists some value v_j corresponding to threshold $\vec{q_j}$ such that $v(S) = v_j = f(\vec{w}(S))$ and $\vec{w}(S) \geq \vec{q_j}$. This means $v'(S) \geq v_j$. Now, if $v'(S) > v_j$, this means that $v'(S) = v_\ell$ where $v_\ell > v_j$, and in particular there exists some vector $\vec{q_\ell} \in Q$ for which $v'(S) = f(\vec{q_\ell}) = v_\ell$ and $\vec{w}(S) \geq \vec{q_\ell}$. This implies that $f(\vec{w}(S)) \geq f(\vec{q_\ell}) = v_\ell > v_j = f(\vec{w}(S))$, a contradiction. □

Furthermore, r-TTGs are a complete representation of coalitional games (albeit with a potentially exponential number of tasks, as per Proposition 3.1).

Proposition 3.2. *Every monotone coalitional game v is monotone r resource-based, with $r \leq n$.*

Proof Sketch. Given a coalitional game $G = \langle N, v \rangle$, we assume that there are n resource types, and let player i possess 1 unit of the i-th resource. The value of a coalition S is determined by the function $f : \{0,1\}^n \to \mathbb{R}_+$ whose value on the resource vector $\vec{e}^S$ (the indicator vector of the coalition S) is simply $v(S)$. The monotonicity of v implies the monotonicity in each dimension of f. □

Combining Propositions 3.1 and 3.2 we conclude that r-TTGs are a complete representation of coalitional games.

4 Computing an Optimal Coalition Structure in r-TTGs

We first turn our attention to the problem of finding an optimal coalition structure for the class of TTGs. The decision variant of the problem can be formalized as follows

Problem 4.1 (OPTTTG)**.** Given a TTG defined by players $N = \{1, \dots, n\}$ with weights $\vec{w} \in \mathbb{Z}_+^n$, and a set of tasks $(q_1, v_1), \dots, (q_m, v_m) \in \mathbb{Z}_+^2$, as well as a a parameter $R \in \mathbb{Z}_+$, is there a coalition structure CS^* over N such that

$$\sum_{C \in CS^*} v(C) \geq R?$$

WVGs are TTGs with one task, so the hardness results by Elkind *et al.* for the coalition structure generation in WVGs immediately imply that Problem 4.1 is NP-complete; whether a PTAS for the OPTTTG problem exists is an open problem, though we conjecture that one does not exist.

4.1 Warmup: Approximation Algorithms for the OPTTTG Problem in WVGs

Let G be a WVG with weights $\vec{w}$ and a threshold q. We partition N into N_- and N_+ such that $i \in N_-$ iff $w_i < q$. In this manner, each player in N_+ can form a winning coalition by himself. Algorithm 1 outputs a $\frac{1}{2}$-approximation to OPTTTG for WVGs; it is similar to the FIRST FIT algorithm used in bin packing [13]; this yields additional benefits: for example, its quality guarantees hold even when players arrive in an on-line manner (a direct translation from the FIRST FIT algorithm). The proof of Theorem 4.2 is similar to the proof for the approximation guarantee of the FIRST FIT algorithm.

Theorem 4.2. *Algorithm 1 yields a $\frac{1}{2}$-approximation to the* OPTTTG *problem, when the game G is a WVG.*

Proof. Let $OPT(G)$ and $NF(G)$ be the optimal revenue and the revenue found by Algorithm 1 for G, respectively; thus $OPT(G)$ is the optimal number of winning coalitions and $NF(G)$ is the number of winning coalitions found by Algorithm 1 for G. First, assume that $N_+ = \emptyset$, i.e. $N = N_-$. We note that in each iteration of Algorithm 1, there is at most one non-winning coalition, and every winning coalition has total weight less than $2q$. We observe that $W = \sum_{i \in N} w_i$ is greater than $q \times OPT(G)$; indeed, let CS^* be an optimal coalition structure, and let CS_+^* be the winning coalitions in CS^*, then

$$W \geq \sum_{S \in CS_+^*} w(S) \geq \sum_{S \in CS_+^*} q = q \times OPT(G)$$

Algorithm 1 NEXT FIT algorithm

$N' \leftarrow N_-$
$S \leftarrow \emptyset$
$CS' \leftarrow \{\}$
while $N' \neq \emptyset$ **do**
  if $w(S) < q$ **then**
    Transfer some player i from N' to S.
  else
    $CS' \leftarrow CS' \cup \{S\}$
    $S \leftarrow \emptyset$
  end if
end while
$CS' \leftarrow CS' \cup \{S\}$
return $CS' \cup \{\{i\} : i \in N_+\}$

By our first observation, we have

$$2q \times NF(G) + q \geq W > q \times OPT(G),$$

hence $NF(G) > \frac{1}{2}OPT(G) - \frac{1}{2}$.

Since $NF(G)$ and $OPT(G)$ are integers, we must have $NF(G) \geq \frac{1}{2}OPT(G)$.

When N_+ is non-empty, we have $NF(G) = NF(G_-) + |N_+|$ and $OPT(G) = OPT(G_-) + |N_+|$; since $NF(G_-) \geq \frac{1}{2}OPT(G_-)$, we have

$$\begin{aligned} NF(G) &= NF(G_-) + |N_+| \\ &\geq \frac{1}{2}OPT(G_-) + \frac{1}{2}|N_+| = \frac{1}{2}OPT(G). \end{aligned}$$

□

We note that Algorithm 1 offers the same approximation ratio even when the value of winning coalitions is not 1 but some arbitrary value $\alpha \in \mathbb{R}_+$.

4.2 The OPTCS Problem in Threshold Task Games

For general TTGs, we make the minimal assumption that the weight of each player is no more than the minimum threshold (i.e. that the value of single players is 0); this is a departure from the framework of Theorem 4.2 where single player coalitions were allowed.

Theorem 4.3. *Let G be a TTG, such that $v(\{i\}) = 0$ for all $i \in N$, and let v^* be the value of the most valuable task in $\mathcal{T}$; then there exists an efficient algorithm that outputs a coalition structure CS^* whose value is at least $\frac{1}{2}(OPT - v^*)$.*

Proof. We sort tasks in $\mathcal{T}$ by their revenue-to-weight ratio: $\frac{v_j}{q_j}$. Let t_{j^*} be the task that maximizes this ratio and has the smallest value; let G^* be a WVG with players having the same weights as in G, a threshold q_{j^*}, and with winning coalitions having the value v_{j^*}. Let CS^* be an optimal coalition structure for G in which coalition C_j completes task t_j (recall that we assume that there are sufficient copies of each task, such that each task is completed by at most one coalition). We run Algorithm 1 on G^*; let $NF(G^*)$ be the value of the coalition structure found by Algorithm 1, we obtain

$$
\begin{aligned}
OPT(G) &\le \sum_{C_j \in CS^*} v_j \frac{w(C_j)}{q_j} \le \frac{v_{j^*}}{q_{j^*}} w(N) \\
&< \frac{v_{j^*}}{q_{j^*}} \left(\frac{NF(G^*)}{v_{j^*}} \times 2q_{j^*} + q_{j^*} \right) \\
&= 2NF(G^*) + v_{j^*}.
\end{aligned}
$$

Hence $NF(G^*) > \frac{1}{2}(OPT(G) - v_{j^*})$. □

As Theorem 4.3 shows, applying Algorithm 1 to TTGs does not yield a true $\frac{1}{2}$-approximation; in fact, playing with only the best task can never yield $\frac{1}{2}OPT(G)$ of the revenue in some cases.

Example 4.4. Consider $6n + 3$ players of weight q, with two tasks having threshold $q + 1$ and $3q$ and value $q + 1 + \epsilon$ and $3q$ respectively. Playing with the best task, i.e., the former one, gives a revenue of $(3n + 1)(q + 1 + \epsilon)$ while the best strategy is to play with the latter one, which yields $(2n + 1)3q$. If q is sufficiently large compared to n, we have

$$
(3n + 1)(q + 1 + \epsilon) < \frac{1}{2}(2n + 1)3q.
$$

Example 4.4 shows that selecting only the best task is not necessarily a good strategy; how does one derive the best set of tasks? Surprisingly, this can be done by returning to the setting initially proposed by Chalkiadakis *et al.*, allowing players to split their weight amongst tasks, followed by an allocation of (unsplit) weights to the best tasks. When players may split weights between tasks, individual weights no longer matter; we simply write $W = \sum_{i \in N} w_i$ and ask: what is the best set of tasks that can be completed, assuming that their total weight cannot exceed W? Formally,

Problem 4.5 (SPLIT-TTG)**.** Given W the total weight of players. Maximize $\sum_{i=1}^{m} x_i v_i$, such that $\sum_{i=1}^{m} x_i q_i \le W$, where $x_i \in \mathbb{Z}_+$ is the variable indicating how many copies of task t_i are completed.

Chalkiadakis *et al.* show that Problem 4.5 can be solved in time polynomial in n and $|W|$, where $|W|$ is the number of bits required to represent W in binary (i.e. a pseudopolynomial time algorithm).

Theorem 4.6 (Chalkiadakis *et al.* 2010)**.** *The* SPLIT-TTG *problem can be solved in time polynomial in $|W|$ and n.*

Leveraging Theorem 4.6, we now present a pseudopolynomial time $\frac{1}{2}$-approximation algorithm for the OPTTTG problem.

Theorem 4.7. *Given an n player TTG G defined by the weight vector $\vec{w}$ and tasks $\mathcal{T}$, such that $v(\{i\}) = 0$ for all $i \in N$, there exists a $\frac{1}{2}$-approximation algorithm for the* OPTTTG *problem, that runs in time polynomial in n and $|W|$.*

Proof. Let T^* be the set of tasks completed by the pseudopolynomial time algorithm by Chalkiadakis *et al.* for the SPLIT-TTG problem. Consider Algorithm 2, whose input is the TTG G and $T^* = (t_1, \dots, t_\ell)$, where the tasks in T^* are assumed to be sorted in decreasing order of their thresholds $q_1 \ge \dots \ge q_\ell$ (thus $v_1 \ge \dots \ge v_\ell$).

We first observe that since all tasks in $\mathcal{T}$ can be completed, there will be at least one coalition in CS' that has a positive value; moreover, CS' contains at most one coalition whose value is 0 at all iterations of Algorithm 2. Suppose that CS' contains k coalitions with nonzero value; we write $CS' = \{S_1, \dots, S_k, S_-\}$ where S_- is a (potentially empty) coalition with value 0. We know that

Algorithm 2 PSEUDO-TTG algorithm

$S \leftarrow \emptyset$
$k \leftarrow 1$
$CS' \leftarrow \{\}$
$N' \leftarrow N$
while $N' \neq \emptyset$ **do**
  if $w(S) < q_k$ **then**
    Transfer some player i from N' to S
  else
    $CS' \leftarrow CS' \cup \{S\}$ ▷ S completed task t_k, move to next task
    $k \leftarrow k + 1$
    $S \leftarrow \emptyset$
  end if
end while
$CS' \leftarrow CS' \cup S$ ▷ S is a losing coalition
return CS'

$|S_j| \geq 2$ for all $j \in \{1, \ldots, k\}$ (as we assume each task requires at least two players to complete). Let i_j be the last player added to S_j; we replace S_j with $U_j = S_j \setminus \{i_j\}, V_j = \{i_j\}$. We have $w(U_j) < q_j$ and $w(V_j) < q_{j+k}$.

If $2k + 1 \leq \ell$, we have $w(N) = \sum_{j=1}^{k} w(U_j) + \sum_{j=1}^{k} w(V_j) + w(S_-) < \sum_{j=1}^{k} q_j + \sum_{j=1}^{k} q_{j+k} + q_{2k+1} \leq \sum_{j=1}^{l} q_j \leq w(N)$; the last inequality holds since we know that the ℓ tasks in T^* (and in particular the first $2k+1$ tasks) can be completed by a fractional allocation of $w(N)$, and we have a contradiction. Therefore, we have $2k + 1 > \ell$ so $k \geq \frac{\ell}{2}$. This means that under CS', the top $\lceil \frac{\ell}{2} \rceil$ most valuable tasks in $\mathcal{T}^*$ are completed. Finally, since the revenue gained by the tasks in $\mathcal{T}^*$ is at least the revenue gained by an optimal (non overlapping) coalition structure, we have that the value of CS' is at least $\frac{1}{2} OPT(G)$. □

Whether a truly polynomial time $\frac{1}{2}$-approximation algorithm for the OPTTTG problem exists is an open problem.

4.3 OPTCS with Multiple Resource Types

Assuming multiple resource types significantly increases problem complexity. However, we can still construct an approximation algorithm whose quality depends exponentially on the number of resources r.

Theorem 4.8. *For a r-WVG, there exists a $\frac{1}{2 \times 3^{r-1}}$-approximation algorithm to compute the optimal coalition structure.*

Proof. We are given an instance of an r-WVG with weight vectors $\vec{w}^1, \ldots, \vec{w}^r$, and quotas $q_1, \ldots, q_r$. We note that we can scale players' weights and quota in every one of the r vectors comprising the r-WVG by the same factor without changing the problem. Therefore, we can assume that all quotas are equal to some $q \in \mathbb{Z}_+$. Moreover, if there exists i, j such that $w_i^j > q$, we can simply replace w_i^j by q. We rearrange the components such that if $j < j'$ then $w^j(N) \leq w^{j'}(N)$. We describe the algorithm as follows:

1. Run Algorithm 1 for the game with players' weights as per $\vec{w}^1$. Distribute all losing players to winning coalitions and let CS_1 be resulting coalition structure.

2. Let $j = 2$.

3. If $j = r + 1$, terminate the algorithm and output CS_{j-1}. Otherwise, go to step 4.

4. We partition the coalitions in CS_{j-1} into two disjoint sets X_j and Y_j such that $w^j(C) < q$ if $C \in X_j$ and $w^j(C) \geq q$ if $C \in Y_j$.
   If $|Y_j| \geq \frac{1}{3}|CS_{j-1}|$, distribute all other players into the coalitions in Y_j to form a coalition structure CS_j; let $j := j + 1$ and go to step 3.
   If $|Y_j| < \frac{1}{3}|CS_{j-1}|$, let $Z_j = \bigcup_{C \in Y_j} C$. Run algorithm (1) for the players in Z_j with their j-th weight component. Let W_j be the set of winning coalitions and L_j be the set of losing coalitions. Let $s = \min\{|W_j|, |X_j|\}$. Select set S of s coalitions in X_j with smallest total weight of j-th component and s coalitions in W_j. Form CS_j consisting of s new coalitions by merging one coalition from X_j and one from W_j.
   If $s \geq \frac{1}{3}|CS_{j-1}|$, distribute all other players into the coalitions in CS_j; let $j := j + 1$ and go to step 3.
   If $s < \frac{1}{3}|CS_{j-1}|$, consider a WVG in which each player is a coalition $C \in L_j \cup (X_j \setminus S)$ with weight $w^j(C)$ and run algorithm (1). Let P_j be the set of winning coalitions; for each set $P \in P_j$ add $\bigcup_{C \in P} C$ to CS_j and distribute all other players into the coalitions in CS_j; let $j := j + 1$ and go to step 3.

We can see that CS_j satisfies $|CS_j| \geq \frac{1}{3}|CS_{j-1}|$. Indeed, we only need to consider the case where in step 4, $s < \frac{1}{3}|CS_{j-1}|$. Let $p = |CS_{j-1}|$, $U = \sum_{C \in CS_{j-1}} w^j(C)$, $U_1 = \sum_{C \in X_j} w^j(C)$, $U_2 = \sum_{C \in W_j} w^j(C)$ and $U_3 = \sum_{C \in L_j} w^j(C)$. Since $|X_j| > \frac{2}{3}p$, we have $s = |W_j|$. We have $U_1 + U_2 + U_3 = w^j(N) \geq w^{j-1}(N) \geq pq$. S is the set of s coalitions in X_j with smallest total weight of j-th component; hence $w^j(X_j \setminus S) \geq \frac{|X_j|-s}{|X_j|}U_1$. When running algorithm (1), there is at most one losing coalition and each winning one weighs less than $2q$. Therefore, we have $s > \frac{1}{2q}U_2$, $|P_j| > \frac{1}{2q}\left(\frac{|X_j|-s}{|X_j|}U_1 + U_3 - q\right)$. Hence $|CS_j| = s + |P_j| > \frac{1}{2q}(U_1 + U_2 + U_3 - \frac{s}{|X_j|}U_1) - \frac{1}{2} = \frac{p}{2} - \frac{U_1 s}{2q|X_j|} - \frac{1}{2}$. Note that $U_1 < q|X_j|$, and because $s < \frac{1}{3}p$ thus $s \leq \frac{1}{3}p - \frac{1}{3}$; we have $|CS_j| > \frac{p}{2} - \frac{1}{2}(\frac{p}{3} - \frac{1}{3}) - \frac{1}{2} = \frac{p}{3} - \frac{1}{3}$. Therefore $|CS_j| \geq \frac{p}{3} = \frac{1}{3}|CS_{j-1}|$. □

5 The Cost of stability in r-TTGs

In what follows, we analyze the cost of stability for r-TTGs; we begin by presenting a tight upper bound when G is a TTG.

Theorem 5.1. *For any TTG G, $CoS(G) \leq 2$. Moreover, the upper bound is tight: for any $\epsilon > 0$ there exists a WVG G such that $CoS(G) > 2 - \epsilon$.*

Proof. For a task t_j, let c_j be the maximum number such that for every coalition C, if $w(C) \geq q_j$ then $w(C) - c_j \geq q_j$. Also, let C_j be the coalition such that the equality holds, i.e. $w(C_j) - c_j = q_j$. We denote by t_{j^*} the task t_j that has the maximum $\frac{v_j}{q_j + c_j}$. Let the payoff to player i be $x_i^* = \frac{v_{j^*} w_i}{q_{j^*} + c_{j^*}}$.

First, we verify that $\vec{x}^*$ is a solution to the program. Indeed, suppose that S completes task t_j (i.e. $v(S) = v_j$); we have:

$$\sum_{i \in S} x_i^* = \sum_{i \in S} \frac{v_{j^*} w_i}{q_{j^*} + c_{j^*}} \geq \frac{v_j}{q_j + c_j} \sum_{i \in S} w_i$$
$$\geq \frac{v_j}{q_j + c_j}(q_j + c_j) = v_j = v(S).$$

Therefore, $\vec{x}^*$ is a feasible solution to (1). Let us now bound the total payoff $x^*(N)$:

$$\frac{v_{j^*}}{q_{j^*}+c_{j^*}}\sum_{i\in N} w_i = \frac{v_{j^*}w(C_{j^*})}{q_{j^*}+c_{j^*}} + \sum_{i\notin C_{j^*}} \frac{v_{j^*}w_i}{q_{j^*}+c_{j^*}}$$
$$\leq v_{j^*} + \sum_{i\notin C_{j^*}} \frac{v_{j^*}w_i}{q_{j^*}}$$

Let G^* be the WVG with players $N \setminus C_{j^*}$ with weights as in G, a threshold q_{j^*}, and with the value of a winning coalition being v_{j^*} rather than 1. As shown in Theorem 4.2, $\sum_{i\in N\setminus C_{j^*}} v_{j^*}w_i < 2q_{j^*}\, OPT(G^*) + v_{j^*}q_{j^*}$. We also note that $OPT(G^*) \leq OPT(G) - v_{j^*}$. Putting this together we have that $x^*(N)$ is at most

$$v_{j^*} + \sum_{i\notin C_{j^*}} \frac{v_{j^*}w_i}{q_{j^*}} < v_{j^*} + \frac{1}{q_{j^*}}(2q_{j^*}OPT(G^*) + v_{j^*}q_{j^*})$$
$$=2v_{j^*} + 2OPT(G^*) \leq 2OPT(G).$$

Therefore $CoS(G) < 2$. For tightness, consider a WVG with $2m-1$ players whose weights are all 1, with $q = m$. The CoS for this game is $2 - \frac{1}{m}$. Consider the constraints in Equation (1) for coalitions of size m: there are a total of $\binom{2m-1}{m}$ such coalitions, and player i appears in exactly $\binom{2m-2}{m-1}$ of them. Summing the inequalities we have

$$\sum_{S:|S|=m}\sum_{i\in S} x_i \geq \binom{2m-1}{m} \iff \binom{2m-2}{m-1}\sum_{i=1}^{n} x_i \quad \geq \binom{2m-1}{m} \iff \sum_{i=1}^{n} x_i \geq 2 - \frac{1}{m}$$

Thus, for a sufficiently large m, $CoS(G)$ is arbitrarily close to 2, and we are done. Any m players form a winning coalition; thus in order for the game to be stable, each player is paid at least $\frac{1}{m}$. The total payment is at least $2 - \frac{1}{m}$; meanwhile, there can be at most 1 winning coalition. Therefore $CoS(G) = 2 - \frac{1}{m}$, which is arbitrarily close to 2. □

The tight bound in Theorem 5.1 assumes that $OPT(G) = 1$; however, when $OPT(G) > 1$ a better bound can be obtained for WVGs, as shown in Theorem 5.2.

Theorem 5.2. *For a WVG G defined by $\langle \vec{w}; q\rangle$, $CoS(G) < \frac{3}{2} + \frac{1}{OPT(G)}$.*

Proof. First we assume that the weight of every player is strictly less than the threshold. Given an optimal coalition structure CS^* for G, let CS^*_+ be the set of winning coalitions and CS^*_- be the set of losing coalitions; thus, $|CS^*_+| = OPT(G)$. Furthermore, we assume that CS^* has only one losing coalition L, and that the total weight of L is maximized. We first observe that if ω is the smallest weight in a coalition $C \in CS^*$, $w(C) < q + \omega$. We partition CS^*_+ into disjoint subsets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, where (a) $\mathcal{A}$ consists of coalitions with total weight of players strictly greater than $\frac{3q}{2}$. A coalition $S \in \mathcal{A}$ consists of exactly two players of weight strictly greater than $\frac{q}{2}$ (otherwise some player in S can be moved to L); (b) $\mathcal{B}$ consists of coalitions with the weight of every player is strictly lower than $\frac{q}{2}$; (c) $\mathcal{C}$ consists of the other coalitions with at least one player (at most two) of weight at least $\frac{q}{2}$. We also abuse notation by calling a winning coalition (not necessarily in CS^*_+) type $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ if it has the required properties.

We can transform CS^* such that either $\mathcal{A}$ or $\mathcal{B}$ is empty. Suppose that both $\mathcal{A}$ and $\mathcal{B}$ are not empty. Let $C_1 \in \mathcal{A}$ and $C_2 \in \mathcal{B}$ be the coalitions that contain the heaviest players i_1 and i_2 among the players in $\mathcal{A}$ and $\mathcal{B}$, respectively. If $w_{i_1} + w_{i_2} \geq q$, we also have $w(C_1 \setminus \{i_1\}) + w(C_2 \setminus \{i_2\}) > \frac{3q}{2} - w_{i_1} + q - w_{i_2} > q$, since $w_{i_1} < q$ and $w_{i_2} < \frac{q}{2}$. Therefore we can replace C_1 and C_2 with $\{i_1, i_2\}$ and $C_1 \setminus \{i_1\} \cup C_2 \setminus \{i_2\}$ both of type $\mathcal{C}$, without changing the optimality of $\mathcal{CS}_*$.

Now let us suppose that $w_{i_1} + w_{i_2} < q$. Let i'_1 be the other player in C_1 . Note that $w_{i'_1} \le w_{i_1} < q - w_{i_2}$. We have $\frac{3q}{2} < w_{i'_1} + w_{i_1} < 2q - 2w_{i_2}$, hence $w_{i_2} < \frac{q}{4}$. Moreover, $w_{i_1} \ge w_{i'_1} > \frac{3q}{2} - (q - w_{i_2}) = \frac{q}{2} + w_{i_2}$. Since w_{i_2} is the heaviest weight in C_2, we can choose a subset C'_2 of C_2 such that $\frac{q}{2} \le w(C'_2) < \frac{q}{2} + w_{i_2}$. We have $w_{i_1} + w(C'_2) > q$ and $w_{i'_1} + w(C_2 \setminus C'_2) > \frac{q}{2} + w_{i_2} + q - (\frac{q}{2} + w_{i_2}) = q$. Hence we can replace C_1 and C_2 by $\{i_1\} \cup C'_2$ and $\{i_2\} \cup C_2 \setminus C'_2$ both of type $\mathcal{C}$, without changing the optimality of $\mathcal{CS}_*$.

In both cases, we reduce $|\mathcal{A}|+|\mathcal{B}|$ by 2. We can continue until either $\mathcal{A}$ or $\mathcal{B}$ is empty. Therefore, we can assume that either $\mathcal{A}$ or $\mathcal{B}$ is empty in CS^*. Consider the two cases:

Case 1 - $\mathcal{A} \neq \emptyset$: Let w^* be the maximum weight in $\mathcal{A}$ and let $C_1 = \{i_1, i'_1\} \in \mathcal{A}$ such that $w_{i_1} = w^*$. Consider the following payoff $\vec{x}^*$:

$$x_i^* = \begin{cases} \frac{w_i}{2(q-w^*)} & \text{if } w_i < q - w^* \\ \frac{1}{2} & \text{if } q - w^* \le w_i \le w^* \\ 1 - \frac{q-w_i}{2(q-w^*)} & \text{otherwise.} \end{cases}$$

If $w_i \le \frac{q}{2}$ then $\frac{w_i}{q} \le \frac{w_i}{2(q-w^*)} \le x_i \le \frac{1}{2}$; on the other hand, if $w_i > \frac{q}{2}$ then $\frac{1}{2} \le x_i \le 1 - \frac{q-w_i}{2(q-w^*)} \le \frac{w_i}{q}$. Consider a winning coalition C. If C has at least 2 players of weight at least $q - w^*$ then $x^*(C) \ge 1$. If every player in C has weight lower than $q - w^*$, we have $x^*(C) \ge \frac{w(C)}{q} \ge 1$. Suppose next that C contains one player i of weight $w_i \ge q - w^*$ whereas other players have a weight lower than $q - w^*$. C receives $x^*(C) \ge \frac{q-w_i}{2(q-w^*)} + x_i^*$. If $w_i \le w^*$ then $x^*(C) \ge \frac{1}{2} + \frac{1}{2} = 1$; otherwise $x^*(C) \ge \frac{q-w_i}{2(q-w^*)} + 1 - \frac{q-w_i}{2(q-w^*)} = 1$. In any case, a winning coalition always receives at least 1, so $\vec{x}^*$ is stable.

Furthermore, no winning coaltion in CS^*_+ should receive more than $\frac{3}{2}$. Indeed, if $C \in \mathcal{A}$, it receives 1; if $C \in \mathcal{C}$ with two players of weight at least $\frac{q}{2}$, C receives $1 \le x(C) \le \frac{w(C)}{q} \le \frac{3}{2}$. If C has exactly one player of weight at least $\frac{q}{2}$, let i of weight w_i be that player and i' of weight $w_{i'}$ be the heaviest player among $C \setminus \{i\}$. If $w_{i'} + w^* \ge q$, there should be no other player in C other than i and i' because otherwise we can form $\{i_1, i'\}$ and $\{i'_1, i\}$ and increase the weight of L. Therefore $x^*(C) = x_i^* + x_{i'}^* < 1 + \frac{1}{2} = \frac{3}{2}$. If $w_{i'} + w^* < q$, $w(C) < q + w_{i'} < 2q - w^*$ then $x^*(C) \le 1 - \frac{q-w_i}{2(q-w^*)} + \frac{w(C)-w_i}{2(q-w^*)} = 1 + \frac{w(C)-q}{2(q-w^*)} < \frac{3}{2}$.

Note that no subset $C \subset L$ can have $q - w^* \le w(C) < w^*$; otherwise we can reform coalitions with i_1 and i'_1 and increase the weight of L. If there exists a player $i \in L$ with $w_i \ge w^*$, we have $x(L) < \frac{q-w_i}{2(q-w^*)} + 1 - \frac{q-w_i}{2(q-w^*)} = 1$. Otherwise, each player must have weight lower than $q - w^*$, and $w(L) < q - w^*$. Hence, $x(L) \le \frac{w(L)}{2(q-w^*)} \le \frac{1}{2} < 1$.

The total payoff is therefore less than $\frac{3}{2}OPT(G) + 1$.

Case 2 - $\mathcal{A} = \emptyset$: We pay each player i an amount $x_i^* = \frac{w_i}{q}$. If C is a winning coalition then $x^*(C) = \frac{w(C)}{q} \ge 1$; therefore $\vec{x}^*$ is stable. Each winning coalition in CS^*_+ contains a player of weight no more than $\frac{q}{2}$; hence $x^*(N) < w(L) + \frac{3q}{2q}OPT(G) < 1 + \frac{3}{2}OPT(G)$.

Now suppose that there are k players whose weight is at least the threshold. We simply pay them 1. The total payoff is bounded by: $k + 1 + \frac{3}{2}(OPT(G) - k) \le 1 + \frac{3}{2}OPT(G)$. Therefore, we always have $CoS(G) < \frac{3}{2} + \frac{1}{OPT(G)}$.

□

While we do not have an upper bound on the cost of stability for r-TTGs, it is straightforward to utilize Theorem 4.8 to bound $CoS(G)$ when G is an r-WVG: i.e. an r-TTG with only one task.

Theorem 5.3. *If G is an r-WVG then* $CoS(G) \le 2 \times 3^{r-1}$.

6 Computing the Shapley value for r-TTGs

Matsui and Matsui (see also [11, Chapter 4]) present a pseudopolynomial algorithm to compute the Shapley value of a WVG. This algorithm can be generalized to compute the Shapley value of any r-TTG, with a running time exponential in the number of resources r.

Theorem 6.1. *There exists a pseudopolynomial time dynamic programming algorithm that computes the Shapley value of a player in an r-TTG G; its running time is polynomial in n and $|w_{\max}|^r$, where $w_{\max}$ is the maximal amount of any resource owned by any player.*

Proof. We assume that player i has the weight vector $\vec{w}_i \in \mathbb{R}^r_+$; since G is an r-TTG, there exists some function $f : \mathbb{R}^r_+ \to \mathbb{R}_+$ such that $v(S) = f(\sum_{i \in S} \vec{w}_i)$ for all $S \subseteq N$. For ease of notation we write $\vec{w}(S) = \sum_{i \in S} \vec{w}_i$. The Shapley value of player i, $\phi_i(G)$, can be rewritten as:

$$\frac{1}{n!} \sum_{s=0}^{n-1} s!(n-1-s)! \sum_{S \subseteq N \setminus \{i\} : |S| = s} v(S \cup \{i\}) - v(S)$$

The inner summation can be then rewritten as:

$$\sum_{\vec{W}=\vec{0}}^{\vec{w}(N)-\vec{w}_i} X_i(\vec{W}, s) \left(f(\vec{W} + \vec{w}_i)) - f(\vec{W}) \right), \tag{2}$$

where $X_i(\vec{W}, s)$ denotes the number of coalitions $S \subseteq N \setminus \{i\}$ of size s such that $\vec{w}(S) = \vec{W}$. We use dynamic programming to compute $X_i(\vec{W}, s)$. For ease of exposition, we present the case where $i = n$ (and write $X_n(\vec{W}, s)$ as $X(\vec{W}, s)$). Let $X[j, \vec{W}, s]$ be the number of s-element subsets of $\{1, \dots, j\}$ that have weight $\vec{W}$ with j ranging from 1 to $n-1$, s from 0 to $n-1$ and $\vec{W}$ is a non-negative vector between $\vec{0}$ to $\vec{w}(N) - \vec{w}_i$. For $s = 0$, $j = 1, \dots, n-1$, we have

$$X[j, \vec{W}, 0] = \begin{cases} 1 & \text{if } \vec{W} = \vec{0} \\ 0 & \text{otherwise.} \end{cases}$$

In general, we have

$$X[j, \vec{W}, s] = X[j-1, \vec{W} - \vec{w}_j, s-1] + X[j-1, \vec{W}, s],$$

assuming that $X[j, \vec{W}, s] = 0$ whenever any of its input parameters is negative. Thus we can compute inductively $X[n-1, \vec{W}, s] = X(\vec{W}, s)$ for all $\vec{W} = \vec{0}, \dots, \vec{w}(N) - \vec{w}_i$ and all $s = 0, \dots, n-1$. Plugging this into Equation (2), we can then compute the Shapley value of player n (and more generally of player i) in r-TTGs. The number of entries of $X[j, \vec{W}, s]$ we need to compute is polynomial in n and $|w_{\max}|^r$, which concludes the proof. □

7 Conclusions and Future Work

In this work we provide efficient (or pseudopolynomial) approximation algorithms for the OPTCS problem in r-TTGs; while our results for WVGs allow players to have singleton winning coalitions, we do require that single players have a value of 0 for the more general class of TTGs. Forgoing this (somewhat minor) assumption remains an open problem for future work. Our $\frac{3}{2} + \frac{1}{OPT(G)}$ bound on the CoS of WVGs is not tight (for example, the game in Example 2.1 has a strictly lower CoS); finding a tighter bound would be an interesting direction for future work. In a recent study, Mash *et al.* present a framework allowing human players to play a WVG; it would be interesting to see how human actors behave in the more general TTG (and r-TTG) setting. Insights from human interaction in a simulated TTG environment would hopefully pave the way to a more informed theory of strategic collaboration in resource-based environments, putting the theory of cooperative games to practice.

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